# Individual Part

**PROBLEM 1:**

Find all square roots of the matrix

$$M = \begin{pmatrix} 4 & 5 & 16 \\ 0 & 9 & 16 \\ 0 & 0 & 25 \end{pmatrix}.$$

(Matrix $B$ is said to be a square root of $M$ if the matrix product $B \cdot B$ is equal to $M$, ie. $B^2 = M$)

**Solution:**

Since $M$ is upper-triangular, its eigenvalues are its diagonal entries, that is, 4, 9 and 25. Let $S$ be a matrix whose columns are eigenvectors of $M$ for the respective eigenvalues 4, 9, 25. Then $S$ has the following form:

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not hard to see that

$$S^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover,

$$D := S^{-1}MS = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

The number of square roots of $M$ is the same as the number of square roots of its diagonalization $D$. It follows from the fact that $E$ is a square root of $M$ if and only if $SES^{-1}$ is a square root of $D$. Let $E$ be a square root of $D$. Then $EE = D$ and hence $ED = DE$. Thus, the latter equality implies that $E$ must preserve eigenspaces of $D$ and consequently $E$ must be diagonal. Hence, we deduce that $D$ has exactly eight square roots, namely

$$\sqrt{D} = \begin{pmatrix} \pm 2 & 0 & 0 \\ 0 & \pm 3 & 0 \\ 0 & 0 & \pm 5 \end{pmatrix}.$$

Hence all solutions to $X^2 = M$ are:

$$\pm \begin{pmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \quad \pm \begin{pmatrix} -2 & 5 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{pmatrix}, \quad \pm \begin{pmatrix} 2 & -5 & 8 \\ 0 & -3 & 8 \\ 0 & 0 & 5 \end{pmatrix}, \quad \pm \begin{pmatrix} 2 & 1 & -8 \\ 0 & 3 & -8 \\ 0 & 0 & -5 \end{pmatrix},$$

□

**PROBLEM 2:**

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic function satisfying $f(x) + f(y) = f(x + y + xy)$ for all $x, y \in \mathbb{R}_+$. Find the value $\dfrac{f(2)}{f(1)}$.

**Solution:**

Put $f(x) = h(1 + x)$, ie. $h(x) = f(x - 1)$. Thus $h : (1, +\infty) \to (0, +\infty)$ and $h(x) + h(y) = f(x - 1) + f(y - 1) = f(xy - 1) = h(xy)$. It is well known fact (see lemma) that $h(x) = \log_\alpha x$ for some $\alpha > 1$. Hence $f(x) = \log_\alpha(1 + x)$ and $\dfrac{f(2)}{f(1)} = \dfrac{\log_\alpha 3}{\log_\alpha 2} = \log_2 3$.

**Lemma** [Functional eq. for logarithm]: *The monotonic functions $f : (1, +\infty) \to \mathbb{R}_+$ satisfying $f(x) + f(y) = f(xy)$ for $x, y > 0$ are of the form $f(x) = \log_\alpha x$ for $\alpha > 1$.*

**Proof of the Lemma:** Putting $y = x$ gives $f(x^2) = 2f(x)$. With $y = kx$ we get $f(x^{k+1}) = (k + 1)f(x)$ for $k = 1, 2, \ldots$. Putting $x = \sqrt[k]{x}$ in the last formula gives $f(x^{1/k}) = \frac{1}{k}f(x)$. Hence we get $f(x^q) = qf(x)$ for every $x > 1$ and rational $q \in \mathbb{Q}_+$.

As function $f$ is monotonic, it has its right and left limits in every point of its domain. So $f(x^r)$ is between $\lim\limits_{n\to\infty} f(x^{p_n}) = \lim\limits_{t\to x_-} f(t^r)$ and $\lim\limits_{n\to\infty} f(x^{q_n}) = \lim\limits_{t\to x_+} f(t^r)$ for any real number $r \in \mathbb{R}_+$ and two sequences of rationals $p_n, q_n \to r$ such that $\ldots < p_n < p_{n+1} < \ldots < r < \ldots < q_{n+1} < q_n < \ldots$. But $f(x^{q_n}) - f(x^{p_n}) = (q_n - p_n)f(x) \longrightarrow 0$, hence $f(x^r) = rf(x)$ for any $x > 1$ and $r > 0$. Now it is enough to take $\alpha = \max\{x \mid f(x) < 1\}$ to have $f(\alpha^t) = t$, ie. $f(x) = \log_\alpha x$. ⋄

□

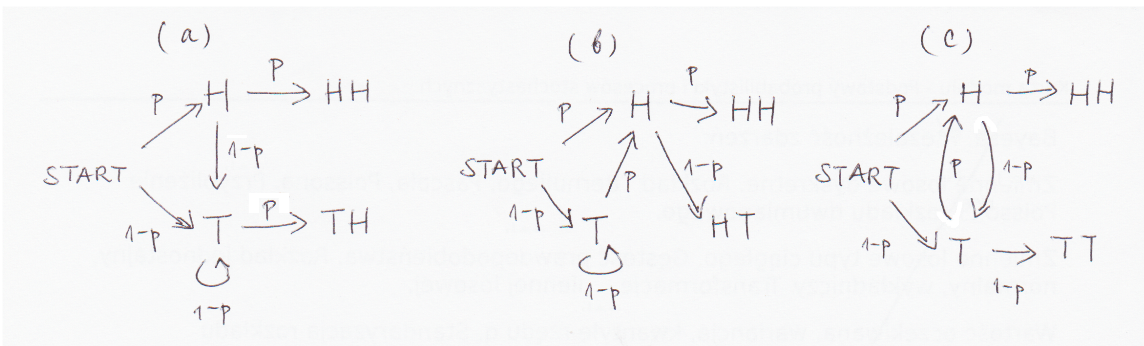
**PROBLEM 3:**

Alice and Ben play a following game: they choose a sequence of two results of coin toss, then they throw a coin until one of those sequences appear. The game is won by whose sequence appeared first.

Ben chose "HH", and before Alice made her move, she has discovered that the coin is unfair: the heads come out with probability 2/3. What sequence should Alice choose to maximize her probability of winning the game? And what is this probability?

**Solution:**

Alice can choose one of the following sequences: (a) "TH", (b) "HT" and (c) "TT". In these cases the game can be represented as the following directed graphs:




So in the case (a) the probability to get "HH" first is only $p^2 = 4/9$, hence "TH" will appear first with probability $1-p^2 = 5/9$.r In the case (b) every game has to pass the point "H", after that the probability to get "HH" is $p = 2/3$, and to get "HT" is $1 - p = 1/3$.

And in the case (c) we have

$$\text{Prob(“HH” first)} = p\Big(p + (1-p)p + p(1-p)p + (1-p)p(1-p)p + \cdots\Big)$$

$$= p\big(p + p(1-p)\big)\sum_{n=0}^{\infty}(1-p)^n p^n$$

$$= \frac{p^2(2-p)}{1-(1-p)p} = \frac{16}{21}$$

To get the probability for "TT" it is enough to interchange $p$ and $1-p$, ie.

$$\text{Prob(“TT” first)} = \frac{(1+p)(1-p)^2}{1-(1-p)p} = \frac{5}{21}$$

In the conclusion Alice should choose "HT", and then her probability to win will be 5/9. □

**PROBLEM 4:**

Calculate the number of subsets of the set $\{0, 1, \ldots, 10\}$ which contain no three consecutive elements.

**Solution:**

Let $C_n$ be the number of subsets of the set $\{0, 1, \ldots, n\}$ with no three consecutive elements. These subsets either contain $n$, or they do not. In the first case either they contain $n-1$ as their element, or they do not. If $n-1$ and $n$ are their element, then they do not contain $n-2$. So their number is a sum of $C_{n-3}$ and $C_{n-2}$.

In the second case their number equals to $C_{n-1}$. Hence

$$C_n = C_{n-1} + C_{n-2} + C_{n-3}\,.$$

Direct computation shows that $C_0 = 2$ ($\varnothing$ and $\{0\}$), $C_1 = 4$ ($\varnothing$, $\{0\}$, $\{1\}$ and $\{0,1\}$) and $C_2 = 7$ ($\varnothing$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, $\{0,2\}$ and $\{1,2\}$). So

$$\begin{pmatrix} C_n \\ C_{n-1} \\ C_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} C_2 \\ C_1 \\ C_0 \end{pmatrix}.$$

Hence $C_{10} = 927$. $\square$

**PROBLEM 5:**

Calculate the integral

$$\int_0^{+\infty} \frac{(1-\cos x)\sin x}{x^2}\,dx\,.$$

**Solution:**

We have

$$\int_0^{+\infty} \frac{(1-\cos x)\sin x}{x^2}dx = \int_0^{+\infty}\Big(\int_0^1 \sin\, xt\,dt\Big)\frac{\sin x}{x}dx = \int_0^1\Big(\int_0^{+\infty} \sin xt\,\sin x\,\frac{dx}{x}\Big)\,dt\,,$$

since all integrand functions are bounded, uniformly continuous and integrable. Let

$$I(w) = \int_0^1\Big(\int_0^{+\infty} \sin xt\,\sin x\,e^{-wx}\,\frac{dx}{x}\Big)\,dt\,.$$

The integral we want to calculate is equal to $I(0)$.

The trigonometric formulas gives $\sin xt\,\sin x = \frac{1}{2}\big(\cos(1-t)x - \cos(1+t)x\big)$. Moreover, for the same reason as before (ie. all functions are bounded, uniformly continuous and integrable), we have

$$\frac{d}{dw}I(w) = -\int_0^1\Big(\frac{1}{2}\int_0^{+\infty}\big(\cos(1-t)x - \cos(1+t)x\big)e^{-wx}\,dx\Big)\,dt\,.$$

The easy integration by parts shows that

$$\int_0^{+\infty}\cos\alpha x\,e^{\beta x}\,dx = \frac{\beta}{\alpha^2+\beta^2}\,.$$

Thus

$$\frac{d}{dw}I(w) = -\frac{1}{2}\int_0^1\Big(\frac{w}{(1-t)^2+w^2} - \frac{w}{(1+t)^2+w^2}\Big)dt = \frac{1}{2}\arctan 2w - \arctan w\,.$$

Now we have $\lim\limits_{w\to+\infty} I(w) = 0$ and $\int \arctan x\,dx = x\,\arctan x - \log\sqrt{1+x^2} + C$. Hence

$$I(0) = -\int_0^{+\infty} I'(w)dw = \lim_{M\to+\infty}\log\frac{\sqrt{1+4M^2}}{\sqrt{1+M^2}} = \log 2\,.$$

□

**Solution 2**:

**Lemma** []: *Let* $f:\mathbb{R}_+\to\mathbb{R}$ *be an integrable real function, such that limits* $\lim\limits_{x\to 0^+} f(x)=f_o$ *and* $\lim\limits_{x\to+\infty} f(x)=f_\infty$ *exist. Then*

$$\int_0^{+\infty}\frac{f(\alpha x)-f(\beta x)}{x}dx=(f_o-f_\infty)\log\frac{\beta}{\alpha}$$

*for any real* $\alpha,\beta>0$.

**Proof of the Lemma:** We have

$$\int_\varepsilon^E\frac{f(\alpha x)}{x}dx=\int_{\alpha\varepsilon}^{\alpha E}\frac{f(x)}{x}dx$$

for $0<\varepsilon<E<+\infty$. Hence

$$\int_\varepsilon^E\frac{f(\alpha x)-f(\beta x)}{x}dx=\int_{\alpha\varepsilon}^{\beta\varepsilon}\frac{f(x)}{x}dx-\int_{\alpha E}^{\beta E}\frac{f(x)}{x}dx=I_\varepsilon-I_E\,.$$

Let now $f(x)=C+o(1)$ with $x\to 0^+$ or $+\infty$, where $o$ is Landau's little-o symbol. Thus

$$\int_{\alpha X}^{\beta X}\frac{f(x)}{x}dx=\int_{\alpha X}^{\beta X}\frac{C+o(1)}{x}dx=\big(C+o(1)\big)\,\log x\Big|_{\alpha X}^{\beta X}=\big(C+o(1)\big)\log\frac{\beta}{\alpha}\,.$$

Hence letting $\varepsilon\to 0^+$ and $E\to+\infty$ leads to

$$\int_0^{+\infty}\frac{f(\alpha x)-f(\beta x)}{x}dx=\lim_{\varepsilon\to 0^+}I_\varepsilon-\lim_{E\to+\infty}I_E=(f_o-f_\infty)\log\frac{\beta}{\alpha}\,.$$

◇

We have

$$\frac{(1-\cos x)\sin x}{x^2}=\frac{\frac{\sin x}{x}-\frac{2\cos x\,\sin x}{2x}}{x}=\frac{f(x)-f(2x)}{x}$$

where $f(x)=\frac{\sin x}{x}$. So, as $\lim\limits_{x\to 0^+}f(x)=1$ and $\lim\limits_{x\to+\infty}f(x)=0$, we get

$$\int_0^{+\infty}\frac{(1-\cos x)\sin x}{x^2}dx=\int_0^{+\infty}\frac{f(x)-f(2x)}{x}dx=(1-0)\log\frac{2}{1}=\log 2\,.$$

□

# Team Part

**PROBLEM 1:**

Calculate the integral

$$\int_0^1 \arctan \frac{x}{1-x} dx.$$

**Solution:**

It is easy to check that $\frac{d}{dx} \arctan \frac{x}{1-x} = \frac{1}{2x^2-2x+1}$. Thus we can compute the integral by integration by parts taking $x - \frac{1}{2}$ as anti-derivative of 1:

$$\int_0^1 1 \cdot \arctan \frac{x}{1-x} dx = (x - \frac{1}{2}) \arctan \frac{x}{1-x}\Big|_0^1 - \int_0^1 \frac{x - \frac{1}{2}}{2x^2 - 2x + 1} dx$$

$$= \lim_{x \to 1^-} (x - \frac{1}{2}) \arctan \frac{x}{1-x} - \frac{1}{4} \ln(2x^2 - 2x + 1)\Big|_0^1 = \frac{\pi}{4}.$$

□

**PROBLEM 2:**

Let $f:(0,\infty)\to(0,\infty)$ be a continuous function with the following properties:

(1) $f(x+y)\leqslant f(x)+f(y)$, for every $x,y\in(0,\infty)$,

(2) $\displaystyle\int_0^\infty \frac{f(x)}{1+x^2}\,dx<\infty$.

Find

$$\lim_{x\to\infty}\frac{f(x)\ln(x)}{x}.$$

**Solution:**

Let $A_n=[2^n,2^{n+1}]$ for every $n\in\mathbb{N}$, let

$$a_n=\sup_{x\in A_n}\frac{f(x)}{x}\quad\text{and}\quad b_n=\inf_{x\in A_n}\frac{f(x)}{x},$$

and let $x_n,y_n\in A_n$ be such that $a_n=\frac{f(x_n)}{x_n}$ and $b_n=\frac{f(y_n)}{y_n}$. Then

$$0\leqslant a_{n+1}=\frac{f(x_{n+1})}{x_{n+1}}=\frac{f(2\frac{x_{n+1}}{2})}{x_{n+1}}\leqslant\frac{2f(\frac{x_{n+1}}{2})}{x_{n+1}}=\frac{f(\frac{x_{n+1}}{2})}{\frac{x_{n+1}}{2}}\leqslant a_n.$$

Moreover

$$\begin{aligned}a_{n+2}=\frac{f(y+x_{n+2}-y_n)}{x_{n+2}}&\leqslant\frac{f(y_n)}{y_n}\frac{y_n}{x_{n+2}}+\frac{f(x_{n+2}-y_n)}{x_{n+2}-y_n}\frac{x_{n+2}-y_n}{x_{n+2}}\leqslant b_n\frac{y_n}{x_{n+2}}+a_n\frac{x_{n+2}-y_n}{x_{n+2}}\\&\leqslant\frac{b_n+7a_n}{8}\end{aligned}$$

We have

$$\int_0^\infty\frac{f(x)}{1+x^2}\,dx\geqslant\int_1^\infty\frac{f(x)}{2x^2}\,dx=\sum_{n=0}^\infty\int_{A_n}\frac{f(x)}{x}\frac{x}{2x^2}\,dx\geqslant\sum_{n=0}^\infty b_n\int_{A_n}\frac{1}{2x}\,dx\geqslant\sum_{n=0}^\infty\frac{b_n\ln(2)}{2},$$

so the series $\sum_{n=0}^\infty b_n$ is convergent. Since $a_{n+2}\leqslant\frac{b_n+7a_n}{8}$,

$$\sum_{n=2}^k a_n\leqslant 7a_0+7a_1+\sum_{n=0}^{k-2}b_n$$

for every $k\geqslant 2$. Hence the series $\sum_{n=0}^\infty a_n$ is convergent. Since the sequence $(a_n)$ is decreasing,

$$2(a_{[\frac{n}{2}]}+\cdots+a_n)\geqslant na_n.$$

On the other hand, since the series $\sum_{n=0}^\infty a_n$ is convergent for every $\varepsilon>0$, there exists $N$ such that $na_n\leqslant\varepsilon$ for every $n\geqslant N$ Hence $\lim_{n\to\infty}na_n=0$. Applying the fact that

$$\frac{f(x)\ln(x)}{x}\leqslant a_n(n+1)\ln(2)$$

for every $x\in A_n$, gives $\lim_{x\to\infty}\frac{f(x)\ln(x)}{x}=0$. □

**PROBLEM 3:**

Determine for which $n = 1, 2, \dots$ there exists a solution to matrix equation $X^2 + X + I_n = 0_n$ in the space $M_n(\mathbb{F}_2)$ of $n\times n$ matrices with entries from the smallest field, ie. $\mathbb{F}_2 = (\{0,1\}, +, \cdot)$ with $0+0 = 1+1 = 0$, $0+1 = 1+0 = 1$ and $0\cdot 0 = 0\cdot 1 = 1\cdot 0 = 0$, $1\cdot 1 = 1$. The matrices $I_n$ and $0_n$ are the identity matrix and the zero matrix of dimension $n\times n$.

**Solution:**

The polynomial $x^2 + x + 1$ has no zeroes in $\mathbb{F}_2$, as $x(x+1) = 0$ for all $x \in \mathbb{F}_2 = \{0,1\}$, so there is no solution in $M_1(\mathbb{F}_2)$. The matrices $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ are the solutions to that equations in $M_2(\mathbb{F}_2)$, what can be verified directly. Hence the equation has solution in $M_{2n}(\mathbb{F}_2)$, namely

$$\begin{pmatrix} V_1 & & \\ & \ddots & \\ & & V_n \end{pmatrix},$$

where $V_1, \dots, V_n$ are the solutions in $M_2(\mathbb{F}_2)$.

In the case of $n$ odd we proceed as following. Let $\mathbb{K}_2$ be the analytic closure of the field $\mathbb{F}_2$, and let $\varepsilon_1, \varepsilon_2 \in \mathbb{K}_2$ be two solutions to $x^2 + x + 1 = 0$. We have $\varepsilon_1 + \varepsilon_2 = 1 = \varepsilon_1 \cdot \varepsilon_2$ and $\varepsilon_1 + \varepsilon_1 = 0 = \varepsilon_2 + \varepsilon_2$ of course.

Now every eigenvalue of the possible solution $X$ should satisfy $x^2 + x + 1 = 0$, so they all are equal to either $\varepsilon_1$ or $\varepsilon_2$. But as number of eigenvalues (with their multiplicity) is odd, there are also odd number of either $\varepsilon_1$ or $\varepsilon_2$ among them, when the number of other eigenvalues are even. Hence the trace of $X$ equals to either $\varepsilon_1$ or $\varepsilon_2$, which is not possible because they are not elements of $\mathbb{F}_2$. $\square$

**PROBLEM 4:**

Let $M$ be a square matrix. Define

$$\cos(M) := I - \frac{1}{2!}M^2 + \frac{1}{4!}M^4 - \frac{1}{6!}M^6 + \cdots,$$

where $I$ is the identity matrix. Calculate $\cos(M)$ for

$$M = \frac{\pi}{2}\begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix}.$$

**Solution:**

Let

$$A = \begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & 7 \end{pmatrix}.$$

Observe that $(1,1)^T$ is an eigenvector for an eigenvalue $\lambda = 4$, but $(1,-1)^T$ is an eigenvector for an eigenvalue $\lambda = 10$. Furthermore,

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Hence

$$\begin{aligned} \cos(M) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2!}\left(\frac{\pi}{2}\right)^2\begin{pmatrix} 4^2 & 0 \\ 0 & 10^2 \end{pmatrix} + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4\begin{pmatrix} 4^4 & 0 \\ 0 & 10^4 \end{pmatrix} - \cdots\right]\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} P \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}, \end{aligned}$$

where

$$P = \begin{pmatrix} \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k)!}(2\pi)^{2k} & 0 \\ 0 & \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k)!}(5\pi)^{2k} \end{pmatrix} = \begin{pmatrix} \cos(2\pi) & 0 \\ 0 & \cos(5\pi) \end{pmatrix}.$$

Finally,

$$\cos(M) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \cos(2\pi) & 0 \\ 0 & \cos(5\pi) \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

□

**PROBLEM 5:**

Let $G$ be a group such that $a^3b^3 = (ab)^3$ and $a^5b^5 = (ab)^5$ for each $a, b \in G$. Show that $G$ is abelian (ie. $ab = ba$ for all $a, b \in G$).

**Solution:**

It is easy to see that

$$a^3b^3(ab)^3 = ababab \Rightarrow a^2b^2 = baba$$

and

$$a^5b^5 = (ab)^5 = ababababab \Rightarrow a^4b^4 = babababa.$$

Both above equalities gives

$$a^2b^2a^2b^2 = a^4b^4 \Rightarrow b^2a^2 = a^2b^2 = baba \Rightarrow ab = ba$$

for all $a, b \in G$. □

**PROBLEM 6:**

Verify wether the number of ways to exchange 10 Euro into coins of value 1, 2, 5 and 10 cents (we assume that two families of coins with the same numbers of coins of the same value coincide) are greater or smaller than 1.5 mln (ie. 1 500 000).

**Solution:**

Let $f_1, f_2, f_5, f_{10} : (-1,1) \to \mathbb{R}$ be given by

$$\begin{aligned}
f_1(x) &= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{1+x+\cdots+x^9}{1-x^{10}} \\
f_2(x) &= \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} = \frac{1+x^2+\cdots+x^8}{1-x^{10}} \\
f_5(x) &= \sum_{n=0}^{\infty} x^{5n} = \frac{1}{1-x^5} = \frac{1+x^5}{1-x^{10}} \\
f_{10}(x) &= \sum_{n=0}^{\infty} x^{10n} = \frac{1}{1-x^{10}} = \frac{1}{1-x^{10}}.
\end{aligned}$$

The number we are looking for is the coefficient $a_{1000}$ of the function

$$\begin{aligned}
g(x) = \sum_{n=0}^{\infty} a_n x^n &= f_1(x) f_2(x) f_5(x) f_{10}(x) \\
&= \frac{(1+x+\cdots+x^9)(1+x^2+\cdots+x^8)(1+x^5)}{(1-x^{10})^4} \\
&= \frac{(1+\cdots+7x^{10}+\cdots+2x^{20}+x^{21}+x^{22})}{(1-x^{10})^4}.
\end{aligned}$$

It is easy to check that

$$h(x) = \frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

for every $|x| < 1$. Gathering together all the facts above gives

$$a_{1000} = 2\binom{98+3}{3} + 7\binom{99+3}{3} + \binom{100+3}{3} = 1\ 712\ 051 > 1500000\,.$$

□

**PROBLEM 7:**

Let $f \in C^1(\mathbb{R})$ be a differentiable function such that it has a second derivative at 0, ie. the limit $f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x}$ does exist. Let now $F : [-1,1] \to \mathbb{R}$ satisfy

$$F(x) = \begin{cases} \dfrac{f(x)-f(0)}{x} & \text{for } x \neq 0 \\ f'(0) & \text{for } x = 0 \end{cases}$$

Calculate $F'(0)$.

**Solution:**

The function $f$ can be represented as Taylor series in Peano's form, ie. with the reminder in Landau's little-o form:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + o(x^2), \qquad \text{as } x \to 0\,.$$

So

$$\frac{F(x)-F(0)}{x} = \frac{1}{2}f''(0) + \frac{o(x^2)}{x^2} \overset{x\to 0}{\longrightarrow} \frac{1}{2}f''(0)\,.$$

□

**PROBLEM 8:**

Let a sequence of functions $f_n : (0,1) \to (0,+\infty)$ be defined as follows:

$$f_0(x) = x \text{ and } f_{n+1}(x) = \sqrt[n+1]{n+1+\big(f_n(x)\big)^{n+1}}.$$

Prove that a sequence $\{f_n\}_{n\geq 0}$ is convergent and $\lim\limits_{n\to\infty} f_n(x) < x + \frac{3+7\sqrt{3}}{6}$.

**Solution:**

First, observe that

$$f_{n+1}(x) = \sqrt[n+1]{n+1+f_n(x)^{n+1}} > \sqrt[n+1]{f_n(x)^{n+1}} = f_n(x).$$

This follows that a sequence $(f_n(x))$ is increasing. In addition, $f_1(x) = 1+x$, $f_2(x) = \sqrt{3+2x+x^2} < \sqrt{3}+x$ and $f_n(x) > \sqrt{3}$ for $n \geq 3$. Since

$$f_{n+1}(x) = \sqrt[n+1]{n+1+f_n(x)^{n+1}},$$

it follows that

$$n+1 = \big(f_{n+1}(x) - f_n(x)\big)\big(f_{n+1}(x)^n + f_{n+1}(x)^{n-1}f_n(x) + ... + f_n(x)^n\big) > \big(f_{n+1}(x) - f_n(x)\big)(n+1)f_n(x)^{n+1}.$$

Consequently, for $n \geq 2$, we obtain

$$f_{n+1}(x) - f_n(x) < \left(\frac{1}{f_n(x)}\right)^n < \left(\frac{1}{\sqrt{3}}\right)^n$$

for $n \geq 2$. Let $n > 2$. Then

$$f_n(x) - f_2(x) = \sum_{k=2}^{n-2}\big(f_{k+1}(x) - f_k(x)\big) < \sum_{k=2}^{+\infty}\left(\frac{1}{\sqrt{3}}\right)^k < \frac{1}{\sqrt{3}(\sqrt{3}-1)} = \frac{3+\sqrt{3}}{6}.$$

Consequently,

$$f_n(x) < x + \sqrt{3} + \frac{3+\sqrt{3}}{6} = x + \frac{3+7\sqrt{3}}{6}.$$

This implies that a sequence $(f_n(x))_{n\geq 0}$ is bounded. Thus, it is convergent and

$$\lim_{n\to\infty} f_n(x) < x + \frac{3+7\sqrt{3}}{6},$$

which completes the solution. $\square$

**PROBLEM 9:**

Let $P : \Omega \to [0, 1]$ be a probability measure defined on the space of elementary events $\Omega$. For any integer $n > 0$ find the minimal number $M(n)$ such that if the sets of events $A_1, A_2, \ldots A_n \subset \Omega$ (measurable with respect to $P$) satisfy $\sum_{i=1}^n P(A_i) > M(n)$, then intersection $A_1 \cap A_2 \cap \cdots \cap A_n$ is non-empty.

**Solution:**

It is easy to see, that if any point of $\Omega$ is in exactly $n-1$ of sets $A_i$, then intersection is empty and the sum of probabilities $\sum_{i=1}^n P(A_i)$ is $n-1$. Thus $M(n) \geq n-1$. We will show equality: suppose, that $\sum P(A_i) > n-1$. Denote $A' = \Omega \setminus A$ complement of $A$]. We have

$$\mu\left(\bigcap A_i\right) = 1 - P\Big(\big(\bigcap A_i\big)'\Big) = 1 - P\left(\bigcup A_i'\right) \geq 1 - \sum P(A_i')$$
$$= 1 - \sum \Big(1 - P(A_i)\Big) = 1 - n + \sum P(A_i) > 1 - n + n - 1 = 0.$$

Thus the intersection has nonzero probability, so it is non-empty. □

**PROBLEM 10:**

Let $X_1$ be a number chosen randomly (with the same probability for each element) from the set $\{0, 1, \dots, 2017\}$. Then we choose randomly a number $X_2 \in \{0, \dots, X_1\}$. And so on, in the same way we choose randomly and independently a number $X_{n+1} \in \{0, \dots, X_n\}$. What is the probability that $\sum\limits_{n=1}^{\infty} X_n < +\infty$?

**Solution:**

We have the conditional mean value $\mathbb{E}(X_{k+1}|X_k) = \frac{1}{2}X_k$ for $k = 1, 2, \dots$, so $\mathbb{E}(X_n) = \mathbb{E}\big(\mathbb{E}(X_n|X_{n-1})\big) = \frac{1}{2}\mathbb{E}(X_{n-1}) = \dots = \frac{1}{2^{n-1}}\mathbb{E}(X_1) = \frac{2017}{2^n}$. Hence

$$\mathbb{E}\Big(\sum_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \frac{2017}{2^n} = 2017 < +\infty\,.$$

This shows that probability that series $\sum\limits_{n=1}^{\infty} X_n$ is infinite equals to 0. So $\mathrm{Prob}\big(\sum\limits_{n=1}^{\infty} X_n < \infty\big) = 1$. □