PROBLEM 1: (VJIMC Ostrava, 2011, problem 2, category II)

Let k be a positive integer. Compute

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\cdots n_k(n_1+\cdots+n_k+1)}\,.$$

PROBLEM 2: (VJIMC Ostrava, 2013, problem 4, category I)

Let n and k be positive integers. Evaluate the following sum

$$\sum_{j=0}^{k}\binom{k}{j}^2\binom{n+2k-j}{2k}.$$

PROBLEM 3: (cf VJIMC Ostrava, 2013, problem 2, category I)

Let $A=(a_{ij})$ and $B=(b_{ij})$ be two real 10×10 matrices such that $a_{ij}=b_{ij}+1$ for all i,j and $A^3=0$. Evaluate $\det B$.

PROBLEM 4: (VJIMC Ostrava, 2017, problem 2, category I)

We say that we extend a finite sequence of positive integers $(a_1,\ldots,a_n)$ if we replace it by

$$(1,2,\ldots,a_1-1,a_1,1,2,\ldots,a_2-1,a_2,1,2,\ldots,a_3-1,a_3,\ldots,1,2,\ldots,a_n-1,a_n),$$

i.e., each element k of the original sequence is replaced by $1,2,\ldots,k-1,k$. Gza takes the sequence $(1,2,\ldots,9)$ and he extends it 2017 times. Then he chooses randomly one element of the resulting sequence. What is the probability that the chosen element is 1?

PROBLEM 5: (VJIMC Ostrava, 2016, problem 2, category I)

Find all positive integers n such that $\varphi(n)$ divides n^2+3.

($\varphi(n)$ denotes Euler's totient function, i.e. the number of positive integers $k\le n$ coprime to n.)

PROBLEM 6: (VJIMC Ostrava, 2014, problem 1, category I)

Find all complex numbers z such that $|z^3+2-2i|+z\bar{z}|z|=2\sqrt{2}$.

($\bar{z}$ is the conjugate of z).

PROBLEM 7: (OMO of BRU Mogilev, 2014, problem 30)

Evaluate the integral

$$\int_0^{+\infty}\frac{x^2-4}{x^2+4}\,\frac{\sin 2x}{x}\,dx\,.$$

PROBLEM 8: (cf. OMO of BRU Mogilev, 2017, problem 30)

Evaluate the integral

$$\int_0^{+\infty}\frac{x-\sin x}{x^3(x^2+4)}dx\,.$$

PROBLEM 9: (cf. OMO of BRU Mogilev, 2017, problem 13)

Let $f(x)=\max\limits_{y\in\mathbb{R}}\big(xy-f(y)\big)$. Find the largest possible value of $f(11)$.

PROBLEM 10: (cf. OMO of BRU Mogilev, 2014, problem 20)

Find the limit

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{\binom{k}{n}}\right)^n.$$

PROBLEM 11: (NCUMC Sankt Petersburg, 2015, problem 3)

Calculate M^{100}, where

$$M = \begin{pmatrix} 1 & 2 & 0 \\ -3 & -3 & 1 \\ 2 & 2 & -1 \end{pmatrix}.$$

PROBLEM 12: (cf NCUMC Sankt Petersburg, 2015, problem 7)

Let A and B be $n \times n$ Hermitian complex matrices such that the list of all non-zero eigenvalues of $A+B$, counted with respect to multiplicities, is exactly the concatenation of the corresponding lists of non-zero eigenvalues of A and B (possibly after reordering). Evaluate the trace $\operatorname{tr}(AB)$.

PROBLEM 13: (cf NCUMC Sankt Petersburg, 2017, problem 2)

Let non-zero function $f : \mathbb{C} \to \mathbb{C}$ satisfy the equality $f(x)f(y) = f(x + e^i ty)$ for fixed $t \in (0, \pi)$ and any $x, y \in \mathbb{C}$. Find the value of $f(i\pi)$.

PROBLEM 14: (NCUMC Sankt Petersburg, 2017, problem 3)

Find the product of all solutions to the equation

$$\sum_{k=1}^{2017} \frac{1}{z - \varepsilon_k} = 0\,,$$

where $\varepsilon_k = e^{ik\pi/1009}$ are different zeroes of the polynomial $z^{2018} - 1$.

PROBLEM 15: (NCUMC Sankt Petersburg, 2016, problem 4)

Calculate the exact value of the integral

$$\int_0^{+\infty} 2^{-x} \frac{2^{x-1} - 1 + 2^{-x-1}}{x^2} dx\,.$$

PROBLEM 16: (NCUMC Sankt Petersburg, 2016, problem 2)

Let $\omega_1, \ldots, \omega_{2016}$ be different roots of degree 2016 of 1. Calculate

$$\prod_{k \neq l} (\omega_k - \omega_l)\,.$$

PROBLEM 17: (Lowell Putnam, 2016, problem B4)

Let A be a $2n \times 2n$ matrix, with entries chosen at random. Each entry is chosen to be 0 or 1, each with probability $\frac{1}{2}$. Find the expected value of $\det(A - A^t)$ as a function of n, where A^t is the transpose of A.

PROBLEM 18: (Lowell Putnam, 2007, problem A3)

Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k{+}1$ are written in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3?

PROBLEM 19: (Lowell Putnam, 2002, problem A5)

Shanille O'Keal shoots free throws on a basket court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots?

PROBLEM 20: (Lowell Putnam, 2015, problem A3)

Compute

$$\log_2 \left(\prod_{a=1}^{2015} \prod_{b=1}^{2015} \left(1 + e^{2\pi i ab/2015}\right) \right).$$